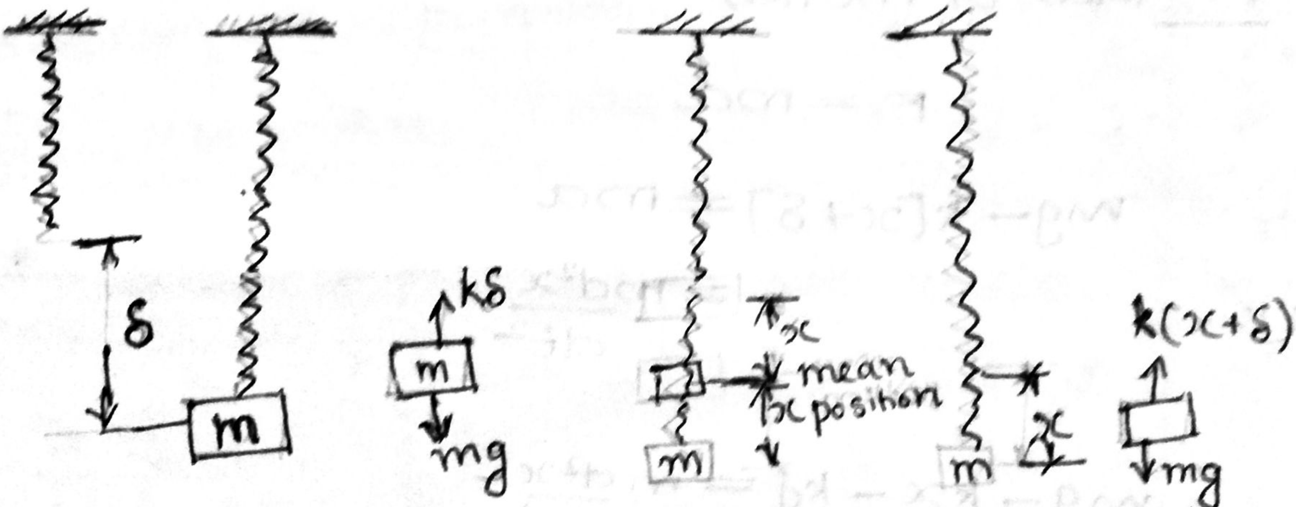
Spring-mass Model:

A mass subjected to a force along the axis of the spring which is attached to the mass is called spring-mass system.

Consider a spring of stiffness 'k', [force required to produce unit deflection] when a body of mass 'm' is attached to one end and with the other end is fixed, the spring elongates. Let 'δ' be the elongation of the spring [static deflection of the spring]



As the system is in static equilibrium the net force acting on the body ='0'. ie,

$$mg - k\delta = 0$$

$$mg = k\delta$$

m = mass of the body,

g = acceleration due to gravity

k = stiffness

δ = deflection

2 when the body is displaced from the equilibrium position by an external force and if the external forces removed then the body will vibrate between two extreme position with amplitude x.

Consider the position of the body when it is at a distance x below the mean position as the body is in motion net force = ma by Newton's second law of motion.

$$F = ma$$

$$mg - k[x+\delta] = ma$$

$$= m\frac{d^2x}{dt^2}$$

$$mg - kx - k\delta = m\frac{d^2x}{dt^2}$$

$$mg - kx - mg = m\frac{d^2x}{dt^2}$$

$$-kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = \frac{-kx}{m}$$

$$\left\{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0\right\}$$

comparing this equation with the equation of SHM

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

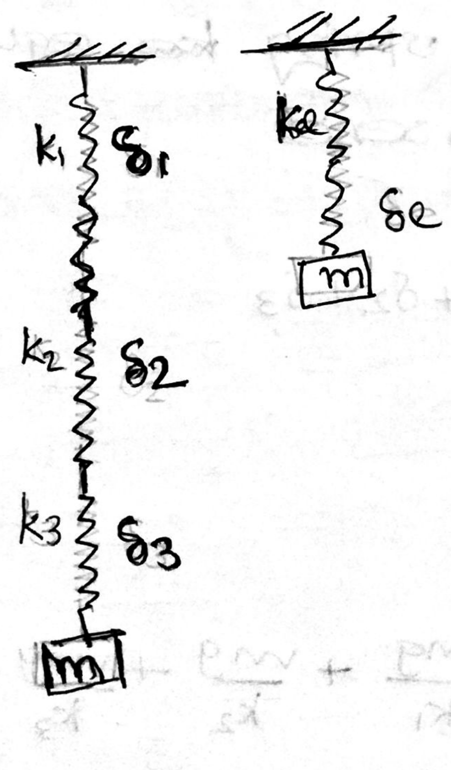
we get $\underline{\omega_n = \sqrt{k/m}}$ (ω_n - natural angular or circular frequency)

$$\boxed{f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{k/m}}$$

fn = natural cyclic frequency of free vibration.

In a spring mass model, the springs can be attached to the mass either in series or in parallel.

Spring in series.



A number of springs having stifness k_1, k_2, etc can be replaced by a single spring of stiffness k_e. the stiffness of the single spring k_e is called <u>equivalent stiffness</u>. Figure shows three springs of stiffness k_1, k_2 and k_3 arranged in series. Let δ_1, δ_2 and δ_3 be the elongation of each spring due to the mass 'm'.

<u>Static deflection of mass</u>

$$\delta = \delta_1 + \delta_2 + \delta_3$$

Let δ_e static deflection of the same mass when it is attached to a single spring of stiffness k_e.

For this single spring ~~ke~~ equivalent to three springs in series

$$\delta_e = \delta$$
$$= \delta_1 + \delta_2 + \delta_3$$

But $mg = k\delta$

$$\delta = \frac{mg}{k}$$

$$\therefore \frac{mg}{k_e} = \frac{mg}{k_1} + \frac{mg}{k_2} + \frac{mg}{k_3}$$

$$\boxed{\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}}$$

Springs in parallel.

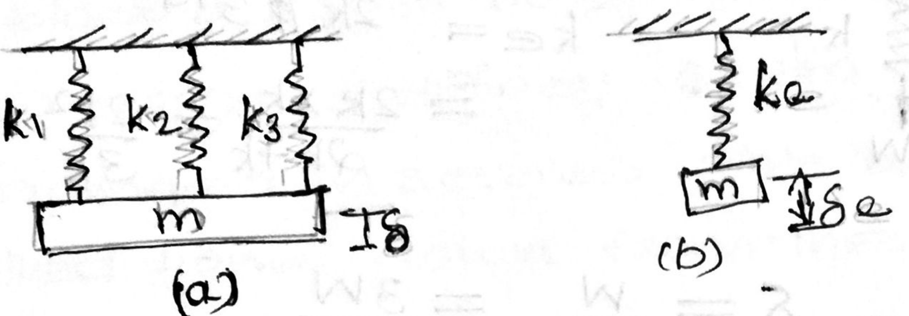


Figure shows three springs arranged in parallel. The required conditions for the single spring to be equivalent to the three springs in parallel

$$\delta_e = \delta$$

For equilibrium of (a)

$$mg = k_1\delta + k_2\delta + k_3\delta$$

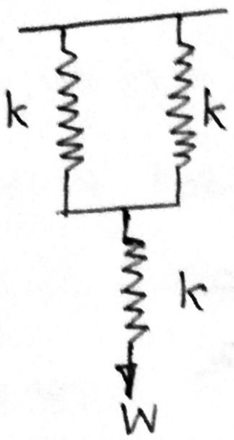
For equilibrium of (b)

$$mg = k_e\delta_e$$

$$k_e\delta_e = \delta[k_1 + k_2 + k_3] \quad [\delta_e = \delta]$$

ie, $\boxed{k_e = k_1 + k_2 + k_3}$

Case 1



consider spring in ∥el

$k_{e1} = 2k.$

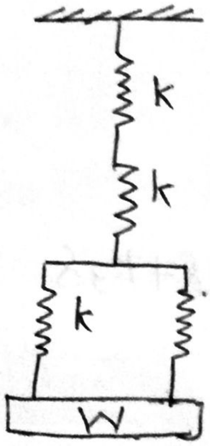
consider all series

$k_e = 2k$ series 3rd

$$= \frac{2k \times k}{2k + k} = \frac{2k^2}{3k} = \frac{2}{3}k$$

$$\delta = \frac{W}{k_e} = \frac{3W}{2k}$$

Case II



For ① & ②

$$k_{e1} = \frac{k^2}{2k} = \frac{k}{2}$$

For ③ & ④

$$k_{e2} = 2k = [k + k]$$

$k_e = k_{e1}$ series k_{e2}

$$= \frac{\frac{k}{2} \times 2k}{\frac{k}{2} + 2k}$$

$$= \frac{k^2}{\frac{5k}{2}}$$

$$= \frac{2k}{5}$$

$$\delta = \frac{W}{K_e} = \frac{5W}{2k}$$

Problems

1. A block of mass 60kg is supported by two springs of stiffness 6kN/m and 8kN/m. arranged in series. The block is pulled 40mm down from the position of equillibrium and then released. Assuming the motion to be SHM. Determine its period of vibration, max velocity and max acceleration.

$k_1 = 6kN/m = 6\times10^3 N/m$

$k_2 = 8kN/m = 8\times10^3 N/m$

$$k = \frac{k_1 k_2}{k_1+k_2}$$

$= 3.42 \times 10^3 N/m$

$= 3428.6 N/m.$

$r = 40\times10^{-3} m.$

$$\omega = \sqrt{k/m} = \sqrt{\frac{k}{60}} = \sqrt{\frac{3428.6}{60}}$$

$= 7.5593$ rad/s

$$T = \frac{2\pi}{\omega} = 0.83 \text{ s}$$

$v_{max} = \omega r = 302.372 \times 10^{-3}$ m/s

$= 0.3$ m/s

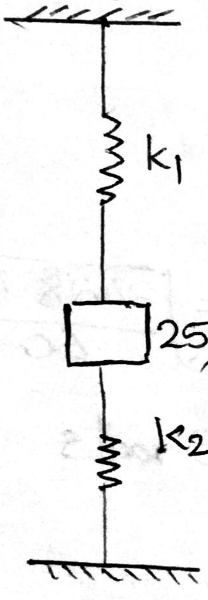
$a_{max} = \omega^2 r = 2.285$ m/s²

2\. Find equivalent spring stiffness and natural frequency of the system in which

a) a 25 kg mass is fixed in between two springs of stiffness $k_1 = 12$ N/mm and $k_2 = 20$ N/mm as shown in fig (a)

b) A 25kg mass is fixed two springs in series with stiffness same as above and the arrangement is shown in (B)

(a)



(b)

a) $k = k_1 + k_2$

Due to mass 25kg spring with stiffness k_1 elongates where as spring with stiffness k_2 contract.

that elongation is equal to contraction.

ie, change in length of both the springs are same. ∴ the given system corresponds to connections in parallel

$$k = k_1 + k_2$$

$$= 12 + 20 = 32 \times 10^3 \text{ N/m}$$

$$\boxed{f = \frac{1}{2\pi}\sqrt{k/m}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{32 \times 10^3}{25}}$$

$$= \cancel{56.19} \text{ Hz}$$

$$5.694 \text{ Hz.}$$

b) Here $k = \frac{k_1 k_2}{k_1 + k_2} = \frac{12 \times 20}{32} = 7.5 \times 10^3 \text{ N/m}$.

$$f = \frac{1}{2\pi}\sqrt{\frac{7.5 \times 10^3}{25}}$$

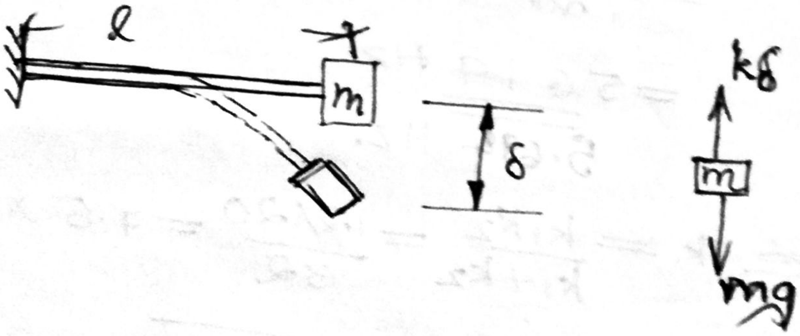
$$= \cancel{27.206} \text{ Hz} \quad 2.756 \text{ Hz}$$

problems also

Angular Vibration

When a body of system oscillates in between two extreme positions such that each and every particle of the vibrating body moves approximately perpendicular to the axis of mounting, the vibration is called angular vibration / transverse vibration. When no periodic forces applied on the vibrating body the vibration is called angular free vibration.

Natural frequency of angular free Vibration



Consider a horizontal beam of stiffness k, Let 'δ' static deflection of a body of mass 'm' attached at the end of the beam as shown. In the static equilibrium condition $mg = k\delta$

we have $\omega = \sqrt{k/m}$ $\quad \therefore k = \frac{mg}{\delta}$

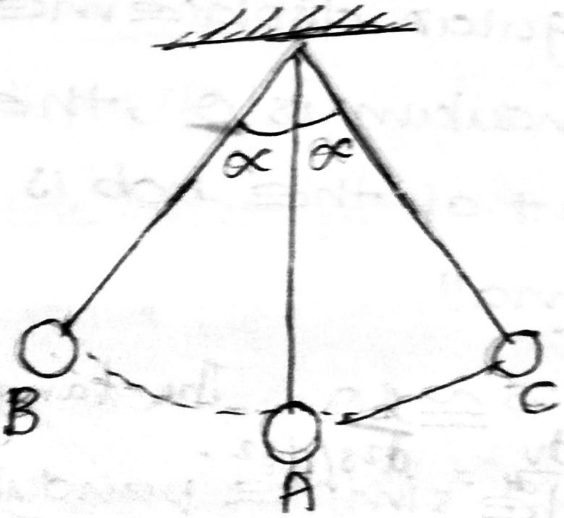
$k/m = g/\delta$

$= \sqrt{g/\delta}$

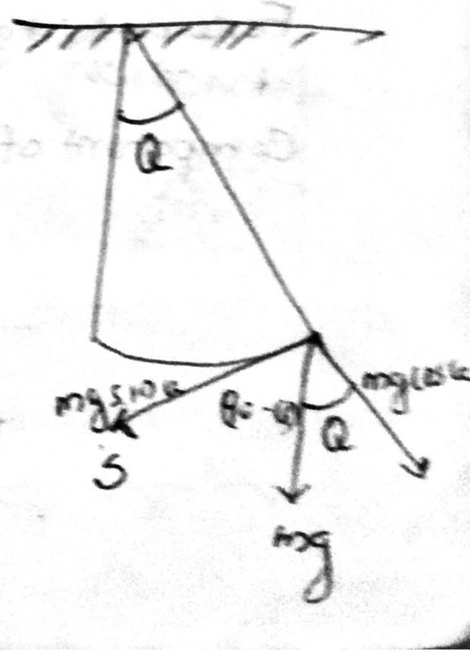
$f_n = \frac{1}{2\pi}\sqrt{g/\delta}$

'δ' depends upon mass of the body 'm', the length of the beam 'l', material of the beam and cross sectional area of beam.

Simple Pendulum.



α – Amplitude of oscillation.



Simple pendulum consist of a bob attached to one end of an inextensible string and the other end of the string is fixed to a rigid support

consider a simple pendulum with length of string l and mass of bob 'm'. Let OA equilibrium position OB and OC are the extreme position

α - amplitude of oscillation

When angular displacement of simple pendulum is θ, the linear displacement of the bob is 's'. when θ is very small

$s \simeq l\theta$. The tangential component of acceln is $\frac{dv}{dt} = d^2s/dt^2$.

~~when the simple pendulum is in motion~~

Force in the tgl direction is the component of gravity force

$$\Sigma f_t = m a_t$$

$a = g\sin\theta$

Component of gravity force in the tangential direction is $(-mg\sin\theta$

$$\Sigma f_t = -mg\sin\theta$$

$$= -mg\theta$$

$$= -mg\frac{s}{l}$$

$$\therefore -mg\, s/l = m a_t$$

$$= m\frac{d^2s}{dt^2}$$

$= -mg\theta$

$= -mg\frac{s}{l}$

ie $\Sigma F = m \times a$

$-mg\frac{s}{l} = m \times \frac{d^2s}{dt^2}$

$\frac{d^2s}{dt^2} + \frac{g}{l}s = 0$

$$\boxed{\frac{d^2s}{dt^2} + \frac{g}{l}s = 0}$$ This is the equation of motion of bob.

comparing this equation with the equation of SHM $\frac{d^2x}{dt^2} + \omega^2 x = 0$, $\omega^2 = \frac{g}{l}$

$$\boxed{\omega = \sqrt{g/l}}$$

$$\boxed{f = \frac{1}{2\pi}\sqrt{g/l}}$$

$$\boxed{T = 2\pi\sqrt{l/g}}$$

The motion of bob from OB to OC is known as a beat.

Time period of 1 beat $= \boxed{\pi\sqrt{l/g}}$

(When the time period of 1 beat is 1 second the pendulum is called second's pendulum.)

Time period of second's pendulum

$= \underline{2 sec}$